**Module 4: Systems of Equations and Matrices**

**II. Systems of Equations in Three Variables**

After completing this section, you should be able to:

* verify whether an ordered triple is a solution of a system of equations in three variables
* solve systems of three linear equations in three variables using Gauss elimination and back-substitution

**A. Solution of a System of Equations in Three Variables**

A **linear equation in three variables** is equivalent to an equation of the form *ax* + *by* + *cz* = *d*, where *a*, *b*, *c*, and *d* are real numbers, and the coefficients *a*, *b*, and *c* are not all 0.

A system of three linear equations in three variables consists of a set of three linear equations.

*a*1*x* + *b*1*y* + *c*1*z* = *d*1  
*a*2*x* + *b*2*y* + *c*2*z* = *d*2  
*a*3*x* + *b*3*y* + *c*3*z* = *d*3

A system of this form is called a **3 × 3 system**, since there are three equations and three variables.

A solution of the system of linear equations is an ordered triple (*x*, *y*, *z*) that satisfies all three of the linear equations.

The graph of a linear equation in three variables is a plane. Graphing planes is beyond the scope of this course, but that doesn't prevent us from obtaining solutions to systems of three linear equations in three variables. In order to solve a 3 × 3 system, an analytical approach will be adopted, similar to the elimination method employed to solve a 2 × 2 system. Just as in the solution of a 2 × 2 system, it turns out that there are three possibilities: one solution, no solution, or infinitely many solutions. A solution (*x*, *y*, *z*) may be verified by checking to see that it satisfies all three equations of the system.

**B. Back-Substitution**

Solving a given 3 × 3 system analytically is generally more complicated than solving a 2 × 2 system. However, if the system of equations has a certain type of "triangular" form, it can be solved quickly, using a technique known as *back-substitution*.

Suppose that a given 3 × 3 system has the following form:

|  |
| --- |
| *x* + *b*1*y* + *c*1*z* = *d*1 *y* + *c*2*z* = *d*2 *z* = *d*3 |

* The first equation involves all three variables *x*, *y*, *z*, and the coefficient of *x* is 1.
* The second equation involves two variables *y* and *z*, and the coefficient of *y* is 1.
* The third equation involves only the variable *z* and coefficient of *z* is 1.

https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M4-Module_4/images/Mod4-7.gifA triangle can be sketched to enclose the expressions on the left side of the equations, and that is why the form is called "triangular."

This 3 × 3 triangular system is easy to solve.

* The third equation already provides the *z*-value of the solution: *z* = *d*3.
* Substitute *z* = *d*3 back into the second equation and then solve for *y*.
* Substitute the *z*-value and the *y*-value back into the first equation and then solve for *x*.

Now the *x*-value, *y*-value, and *z*-value have all been determined.

Triangular systems in which the leading coefficient in each equation is 1 have exactly one solution—one ordered triple (*x*, *y*, *z*)—satisfying all three equations.

**Example II.B.1:** Solve the following 3 × 3 system using back-substitution.

*x* – 2*y* – *z* =   4  
*y* + 3*z* = –1  
*z* = –2

**Solution:**

The value of *z* is already given. Use back-substitution to find *y* and then *x*.

|  |  |
| --- | --- |
| *y* +     3*z*= –1 | Use the second equation. |
| *y* + 3(–2) = –1 | Substitute for *z*. |
| *y* = 5 | Solve for *y*. |
|  | | |
| *x* –   2*y* –*z*= 4 | Use the first equation. |  |
| *x* – 2(5) – (–2) = 4 | Substitute for *y* and *z*. |  |
| *x* – 8 = 4 | Simplify. |  |
| *x* = 12 | Solve for *x*. |  |

The solution (*x*, *y*, *z*) = (12, 5, –2). The solution may be verified by checking to see that it satisfies all three equations.

**C. Gauss Elimination**

Most 3 × 3 systems do not start out in triangular form. Therefore, to solve a 3 × 3 system, the goal is to use an elimination method to transform the equations into an equivalent set of equations in triangular form, and then use back-substitution to find the solution.

Gauss elimination is a systematic elimination method, named in honor of the renowned mathematician Carl Friedrich Gauss, who is mentioned in the overview of this module.

In Gauss elimination, there are three operations that are employed to manipulate equations:

1. **Interchange:** Switch two equations.
2. **Scale:** Multiply both sides of an equation by a nonzero scalar (a nonzero constant).
3. **Replace:** Replace an equation by the sum of the equation and a nonzero multiple of another equation. For example, you could replace the 2nd equation by [5 · (1st equation) + 2nd equation].

To see how the process works, consider the following example.

**Example II.C.1:** Solve the following 3 × 3 system.

|  |
| --- |
| 2*x* + 5*y* + 4*z* = 9 *x* +   *y*+   *z* = 4 *x* –   *y* –   *z* = 6 |

**Solution:**

The goal is manipulate this system and arrive at an equivalent triangular system with leading coefficients of 1, and then use back-substitution to determine the solution.

In the first equation, the coefficient of *x* is 2. However, if the first and second equations are interchanged, then the topmost equation will be in the desired form, with a leading coefficient of 1.

**Step 1:** Get a leading *x*-coefficient of 1 in the first equation.

|  |  |  |
| --- | --- | --- |
| 2*x* + 5*y* + 4*z* = 9 | Interchange 1st and 2nd equations. → | *x* +   *y* +   *z* = 4 |
| *x* +   *y*+   *z* = 4 | 2*x* + 5*y* + 4*z* = 9 |
| *x* –   *y* –   *z* = 6 | *x* – *y* –   *z* = 6 |

The first equation does not require any further modification. However, in order to arrive at triangular form, *x* must be eliminated from the other two equations.

**Step 2:** Eliminate *x* from the second equation.

In the second equation, the coefficient of *x* is 2. If the first equation is multiplied by –2 and added to the second equation, then *x* is eliminated.

|  |  |  |
| --- | --- | --- |
| *x* +   *y* +   *z* = 4 | Replace2nd equation by [(–2) · 1st + 2nd]. → | *x* + *y* +  *z* = 4 |
| 2*x* + 5*y* + 4*z* = 9 | 3*y* + 2*z* =1 |
| *x* –   *y* –   *z* = 6 | *x* – *y* –  *z* = 6 |

|  |  |
| --- | --- |
| Scratch work: | |
| –2*x* – 2*y* – 2*z* = –8 | Multiply 1st equation by –2. |
| 2*x* + 5*y* + 4*z* =   9 | 2nd equation |
| 3*y* + 2*z* =   1 | Add. |

**Step 3:** Eliminate *x* in the third equation.

In the third equation, the coefficient of *x* is 1. If the first equation is multiplied by –1 and added to the third equation, then *x* is eliminated.

|  |  |  |
| --- | --- | --- |
| *x* + *y* +   *z* = 4 | Replace 3rd equation by [(–1) · 1st+ 3rd ]. → | *x* + *y* + *z* = 4 |
| 3*y* + 2*z* = 1 | 3*y* + 2*z* = 1 |
| *x*– *y* –   *z* = 6 | –2*y* – 2*z* = 2 |

|  |  |
| --- | --- |
| Scratch work: | |
| –*x* – *y* –   *z* = –4 | Multiply 1st equation by –1. |
| *x* – *y* –   *z* =   6 | 3rd equation |
| –2*y* – 2*z* =   2 | Add. |

Now *x* has been eliminated in all but the first equation.

Next, manipulate the system so that one of the equations involves only *y* and *z*, and the leading coefficient (the coefficient of *y*) is equal to 1.

You can choose to work with the second equation or the third equation.

In the second equation, the coefficient of *y* is 3. If the second equation is multiplied by 1/3, the resulting equation will have a leading coefficient of 1, a *z*-coefficient of 2/3, and a constant of 1/3 on the right side of the equation.

In the third equation, the coefficient of *y* is –2. If the third equation is multiplied by –1/2, the resulting equation will have a leading coefficient of 1, a *z*-coefficient of 1, and a constant of –1 on the right side of the equation.

Because it is easier to perform arithmetic involving integers rather than fractions, it is preferable to work with the third equation.

**Step 4:** Get a leading *y*-coefficient of 1 in the third equation.

|  |  |  |
| --- | --- | --- |
| *x* + *y* + *z* = 4 | Scale 3rd equation, multiplying by –1/2. → | *x* + *y* +   *z* =   4 |
| 3*y* + 2*z* = 1 | 3*y* + 2*z* =   1 |
| –2*y* – 2*z* = 2 | *y* +   *z* = –1 |

In the triangular form, the second equation should have a *y*-coefficient of 1, so switch the last two equations.

**Step 5:** Get a leading *y*-coefficient of 1 in the second equation.

|  |  |  |
| --- | --- | --- |
| *x* + *y* +   *z* =   4 | Interchange 2nd and 3rd equations. → | *x* + *y* +   *z* =   4 |
| 3*y* + 2*z* =   1 | *y* +   *z* = –1 |
| *y* +   *z* = –1 | 3*y* + 2*z* =   1 |

The second equation does not require any further modification. However, in order to arrive at triangular form, *y* must be eliminated from the third equation.

**Step 6:** Eliminate *y* in the third equation.

In the third equation, the coefficient of *y* is 3. If the second equation is multiplied by –3 and added to the third equation, then *y* is eliminated.

|  |  |  |
| --- | --- | --- |
| *x* + *y* +   *z* =   4 | Replace 3rd equation by [(–3) · 2nd + 3rd]. → | *x* + *y* + *z* =   4 |
| *y* +   *z* = –1 | *y* + *z* = –1 |
| 3*y* + 2*z* =   1 | –*z* =   4 |

|  |  |
| --- | --- |
| Scratch work: |  |
| –3*y* – 3*z* = 3 | Multiply 2nd equation by –3. |
| 3*y* + 2*z* = 1 | 3rd equation |
| –*z* = 4 | Add. |

Now manipulate the system so that the third equation has a leading coefficient equal to 1.

**Step 7:** Get a leading *z*-coefficient of 1 in the third equation.

|  |  |  |
| --- | --- | --- |
| *x* + *y* + *z* = 4 | Scale 3rd equation, multiplying by –1. → | *x* + *y* + *z* = 4 |
| *y* + *z* = –1 | *y* + *z* = –1 |
| –*z* = 4 | *z* = –4 |

The system now has the desired triangular form, with leading coefficients of 1.

**Step 8:** Use back-substitution to determine the solution of the triangular system.

According to the third equation, *z* = –4. Use back-substitution to find *y* and then *x*.

|  |  |
| --- | --- |
| *y* + *z* = –1 | Use the second equation. |
| *y* + (–4) = –1 | Substitute for *z*. |
| *y* = 3 | Solve for *y*. |
|  |  |
| *x* +   *y* +     *z*= 4 | Use the first equation. |
| *x* + (3) + (–4) = 4 | Substitute for *y* and *z*. |
| *x* = 5 | Solve for *x*. |

The solution (*x*, *y*, *z*) = (5, 3, –4). The solution may be verified by checking to see that it satisfies all three equations.

Is it always possible to transform a 3 × 3 system into triangular form with leading coefficients of 1? The answer is no. Just as for a 2 × 2 system, the elimination method may result in an equation of the form 0 = *c* or 0 = 0.

**Example II.C.2:** Solve the following 3 × 3 system.

–*x* + 2*y* + 4*z* = 1  
–2*x* + 4*y* + 9*z* = 4  
3*x* – 6*y* – 8*z* = 5

**Solution:**

None of the equations has a leading coefficient of 1. However, in the first equation, the coefficient of *x* is –1. If the first equation is multiplied by –1 then the topmost equation will be in the desired form, with a leading coefficient of 1.

**Step 1:** Get a leading *x*-coefficient of 1 in the first equation.

|  |  |  |
| --- | --- | --- |
| –*x* + 2*y* + 4*z* = 1 | Scale 1st equation, multiplying by –1. → | *x* – 2*y* – 4*z* = –1 |
| –2*x* + 4*y* + 9*z* = 4 | –2*x* + 4*y* + 9*z* =   4 |
| 3*x* – 6*y* – 8*z* = 5 | 3*x* – 6*y* – 8*z* =   5 |

The first equation does not require any further modification. Now eliminate *x* from the other two equations.

**Step 2:** Eliminate *x* from the second equation.

|  |  |  |
| --- | --- | --- |
| *x* – 2*y* – 4*z* = –1 | Replace 2nd equation by [2 · 1st + 2nd]. → | *x* – 2*y* – 4*z* = –1 |
| –2*x* + 4*y* + 9*z* =   4 | *z* =   2 |
| 3*x* – 6*y* – 8*z* =   5 | 3*x* – 6*y* – 8*z* =   5 |

|  |  |
| --- | --- |
| Scratch work: |  |
| 2*x* – 4*y* – 8*z* = –2 | Multiply 1st equation by 2. |
| –2*x* + 4*y* + 9*z* =   4 | 2nd equation |
| *z* =   2 | Add. |

**Step 3:** Eliminate *x* from the third equation.

|  |  |  |
| --- | --- | --- |
| *x* – 2*y* – 4*z* = –1 | Replace 3rd equation by [(–3) · 1 st + 3 rd]. → | *x* - 2*y* - 4*z* = –1 |
| *z* =   2 | *z* =   2 |
| 3*x* – 6*y*– 8*z* =   5 | 4*z* =   8 |

|  |  |
| --- | --- |
| Scratch work: |  |
| –3*x* + 6*y* + 12*z* = 3 | Multiply 1st equation by –3. |
| 3*x* – 6*y* –   8*z* = 5 | 3rd equation |
| 4*z* = 8 | Add. |

Now *x* has been eliminated in all but the first equation.

Notice that neither the second nor the third equation involves *y*. Therefore, in the triangular form, there will not be an equation involving *y* and *z* with a *y*-coefficient of 1. This is an indicator that the system does not have a unique solution. Whether there is no solution or infinitely many solutions is still to be determined.

The second and third equations involve only the variable *z*. To eliminate *z* from the third equation, multiply the second equation by –4 and add to the third equation.

**Step 4:** Eliminate *z* from the third equation.

|  |  |  |
| --- | --- | --- |
| *x* – 2*y* – 4*z* = –1 | Replace 3rd by [ (–4) · 2nd + 3rd]. → | *x* – 2*y* – 4*z* = –1 |
| *z* =   2 | *z* =   2 |
| 4*z* =   8 | 0 =   0 |

|  |  |
| --- | --- |
| Scratch work: |  |
| –4*z* = –8 | Multiply 2nd equation by –4. |
| 4*z* =   8 | 3rd equation |
| 0 = 0 | Add. |

Regardless of the values of *x*, *y*, and *z*, 0 is equal to 0. Just as in the case of the 2 × 2 system, the equation 0 = 0 indicates that this 3 × 3 system has infinitely many solutions.

**Step 5:** Use back-substitution to determine the solutions.

According to the second equation, *z* = 2.

|  |  |  |
| --- | --- | --- |
| *x* – 2*y* –   4*z* | =       –1 | Use the first equation. |
| *x* – 2*y* – 4(2) | =       –1 | Substitute for *z*. |
| *x* – 2*y* | =         7 | Simplify. |
| *x* | = 2*y* + 7 | Solve for *x*. |

Notice that *x* depends on *y*. There are no restrictions on *y*; it can have any real value.

The solutions of the system are the ordered triples (*x*, *y*, *z*) = (2*y* + 7, *y*, 2), where *y* is any real number.

For instance,

* for *y* = 0, the ordered triple (2*y* + 7, *y*, 2) = (2(0) + 7, (0), 2) = (7, 0, 2) is a solution, and
* for *y* = –3, the ordered triple (2*y* + 7, *y*, 2)= (2(–3) + 1, (–3), 2) = (–5, –3, 2) is a solution.

The previous example illustrates a case in which a 3 × 3 system has infinitely many solutions. In the example, Gauss elimination leads to an equation of the form 0 = 0, an equation that is always true for any variables. If instead, the resulting equation had been 0 = 1, an equation that is never true, then the system would be inconsistent and have no solution. A general rule follows.

If Gauss elimination results in an equation of the form 0 = *c*, where *c* is a nonzero constant, then the system of linear equations is inconsistent and has no solution.

If Gauss elimination does not result in an equation of the form 0 = *c*, where *c* is a nonzero constant, but does result in an equation of the form 0 = 0, then the system of linear equations is consistent and has infinitely many solutions.

Gauss elimination is a powerful method for solving systems of equations. It can be applied to large systems of more than three variables. For example, Gauss elimination can be used to solve a system of 100 linear equations in 100 variables! Computers are used to implement Gauss elimination for large systems. In this course, you will only be asked to solve "small" systems by hand.

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